BPT-Berkeley ALL PROBLEMS

Society for Physics Students, Berkeley

2026

Below is the complete set of written problems as compiled on April 21, 2026. Problems marked with [INCOMPLETE] or [UNDER-REVIEW] are not to be presented on competition day. Each section will be accompanied with a page-break, so printing out this document will provide a whole copy of the test for one student.

None

1 TEAM FRQ

Question 01

Not for Those In Glass Houses

Imagine that you have a rock, with a point mass m.

(a) If you throw this rock from the ground at an angle θ above the horizontal, with an initial velocity v, what is its initial horizontal velocity? Give your answer in terms of v and θ.

(b) If you throw the rock while on flat ground, calculate the total amount of time the rock will spend in the air before first contact with the ground. Assume that the gravitational acceleration is g.

(c) Suppose the rock is subjected to a constant horizontal acceleration a_0 in the forward direction. The vertical motion remains unchanged. Find the horizontal velocity $v_x(t)$ as a function of time.

(d) Calculate the total horizontal distance traveled by the rock with all the variables defined above.

Solution

(a) The horizontal component of the initial velocity is

$$\boxed{v_x(0) = v\cos\theta}$$

(b) The vertical position is

$$y(t) = v\sin\theta\, t - \frac{1}{2}gt^2$$

The projectile returns to the ground when $y(t) = 0$, so

$$t\left(v\sin\theta - \frac{1}{2}gt\right) = 0$$

Ignoring the $t = 0$ launch time, the flight time is

$$\boxed{t_{\text{air}} = \frac{2v\sin\theta}{g}}$$

(c) A constant horizontal acceleration a_0 gives

$$\boxed{v_x(t) = v\cos\theta + a_0 t}$$

(d) Using $t_{\text{air}} = \dfrac{2v\sin\theta}{g}$, the horizontal displacement is

$$\Delta x = v\cos\theta\, t_{\text{air}} + \frac{1}{2}a_0 t_{\text{air}}^2$$

Substituting gives

$$\begin{aligned}\Delta x &= v\cos\theta\left(\frac{2v\sin\theta}{g}\right) + \frac{1}{2}a_0\left(\frac{2v\sin\theta}{g}\right)^2 \\ &= \frac{2v^2\sin\theta\cos\theta}{g} + \frac{2a_0v^2\sin^2\theta}{g^2}\end{aligned}$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\boxed{\Delta x = \frac{v^2\sin 2\theta}{g} + \frac{2a_0v^2\sin^2\theta}{g^2}}$$

Answer

(a) $v\cos\theta$

(b) $\dfrac{2v\sin\theta}{g}$

(c) $v\cos\theta + a_0 t$

(d) $\dfrac{v^2\sin 2\theta}{g} + \dfrac{2a_0v^2\sin^2\theta}{g^2}$

Question 02

Let's get this par-TEA started

Suppose we have two identical cups. Cup A contains hot tea at $T_h = 100°\text{C}$, and Cup B contains pure water at $T_c = 0°\text{C}$. Both cups contain the same mass of liquid and have the same constant heat capacity C. We wish to heat the pure water in Cup B using the thermal energy in Cup A. The two liquids cannot be mixed. Assume that the cups have negligible heat capacity, and that no heat escapes into the environment.

(a) If Cup A and Cup B are placed in thermal contact until they reach equilibrium, what is the final temperature of the pure water?

(b) Suppose we instead divide the hot tea from Cup A into two equal portions. We first bring the first portion into thermal contact with Cup B until equilibrium is reached, then remove it. We then bring the second portion (still at 100°C) into contact with Cup B. What is the final temperature of the pure water? Explain why this result is different than the result in part (a).

(c) If we divide the hot tea into n equal portions and repeat this sequential process - bringing each portion into contact with Cup B one by one and removing it after equilibrium - the final temperature of the pure water approaches a limit as $n \to \infty$. Calculate this upper limit.

(d) Even in the limit $n \to \infty$, why is it impossible to heat the pure water to 100°C using this method?

Solution

Let each full cup of water have the same heat capacity C (mass $\times$ specific heat). The absolute temperature scale does not matter here, so we can work directly with $^\circ$C since only differences enter.

(a) We put the hot cup at $T_h = 100^\circ\text{C}$ in thermal contact with the cold pure water at $T_c = 0^\circ\text{C}$ until equilibrium. By energy conservation,

$$C(T_h - T_f) = C(T_f - T_c),$$

so

$$T_h + T_c = 2T_f \quad \Rightarrow \quad T_f = \frac{T_h + T_c}{2} = \frac{100^\circ\text{C} + 0^\circ\text{C}}{2} = 50^\circ\text{C}$$

Thus the final temperature of the pure water is 50°C.

(b) Now divide the hot cup into two equal parts, each of heat capacity $C/2$. *Step 1:* First half in contact with the pure water. Let the common equilibrium temperature be T_1. Energy conservation gives

$$\frac{C}{2}(T_h - T_1) = C(T_1 - T_c)$$

Rearranging,

$$T_h - T_1 = 2(T_1 - T_c) \ \Rightarrow \ T_h + 2T_c = 3T_1 \ \Rightarrow \ T_1 = \frac{T_h + 2T_c}{3}$$

For $T_h = 100^\circ\text{C}$ and $T_c = 0^\circ\text{C}$, $T_1 = \frac{100}{3}^\circ\text{C} \approx 33.3^\circ\text{C}$. *Step 2:* Now take the second hot half (still at T_h) and bring it into contact with the pure water (now at T_1). Let the new final temperature be T_2. Again,

$$\frac{C}{2}(T_h - T_2) = C(T_2 - T_1),$$

so

$$T_h - T_2 = 2(T_2 - T_1) \ \Rightarrow \ T_h + 2T_1 = 3T_2 \ \Rightarrow \ T_2 = \frac{T_h + 2T_1}{3}$$

Substitute $T_1 = (T_h + 2T_c)/3$:

$$T_2 = \frac{T_h + 2\left(\frac{T_h + 2T_c}{3}\right)}{3} = \frac{5T_h + 4T_c}{9}$$

For $T_h = 100^\circ\text{C}$ and $T_c = 0^\circ\text{C}$,

$$T_2 = \frac{500}{9}^\circ\text{C} \approx 55.6^\circ\text{C}$$

Explanation: This result is hotter than part (a) because the heat transfer is more efficient. In part (a), a lot of heat is transferred across a large temperature difference (which is highly irreversible). By splitting the hot water, the first half does the "heavy lifting" of warming the cold water. The second half is then exposed to water that is *already warm*, meaning it transfers its heat at a higher average temperature, pushing the final equilibrium point higher.

(c) Now divide the hot cup into n equal parts, each with heat capacity C/n. Let T_k be the pure-water temperature after the k-th portion has been brought into contact and equilibrium reached ($k = 0, 1, \ldots, n$), with $T_0 = T_c$ and each hot portion starting at T_h. During the k-th step, the energy balance is

$$\frac{C}{n}(T_h - T_k) = C(T_k - T_{k-1})$$

Cancelling C and rearranging,

$$\frac{1}{n}(T_h - T_k) = T_k - T_{k-1} \ \Rightarrow \ T_h + nT_{k-1} = (n+1)T_k \ \Rightarrow \ T_k = \frac{T_h + nT_{k-1}}{n+1}$$

This is a linear recurrence of the form $T_k = aT_{k-1} + b$ with

$$a = \frac{n}{n+1}, \qquad b = \frac{T_h}{n+1}$$

Solving with $T_0 = T_c = 0^\circ\text{C}$,

$$T_k = T_h\left(1 - a^k\right) = T_h\left(1 - \left(\frac{n}{n+1}\right)^k\right)$$

After all n pieces have been used, the final temperature is

$$T_n = T_h \left(1 - \left(\frac{n}{n+1}\right)^n\right)$$

In the limit $n \to \infty$,

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = e^{-1}$$

So the highest possible final temperature (upper limit) is

$$T_{\max} = \lim_{n\to\infty} T_n = T_h \left(1 - e^{-1}\right)$$

With $T_h = 100°\text{C}$,

$$T_{\max} = 100 \left(1 - e^{-1}\right)^{\circ} \text{C} \approx 63.2°\text{C}$$

(d) We cannot heat the pure water all the way to 100°C because the hot water is a *finite* heat source with the same total heat capacity as the cold water. As it transfers heat, the hot source must cool down. By conservation of energy, the total energy of the system is constant. If Cup B reached 100°C, it would have gained $100C$ of energy, meaning Cup A must have lost $100C$ of energy, bringing it down to 0°C. This would require heat to flow from a colder body to a hotter body, violating the Second Law of Thermodynamics. The theoretical maximum temperature achievable without external work is $T_h(1-e^{-1}) < T_h$.

Answer

(a)

$$T_f = 50°\text{C}$$

(b)

$$T_2 = \frac{500}{9}^{\circ}\text{C} \approx 55.6°\text{C}$$

The result is hotter because sequential contact allows the second half of the hot water to transfer its heat to already-warmed water, retaining a higher temperature potential and making the overall process more efficient (less irreversible).

(c)

$$T_{\max} = 100(1 - e^{-1})°\text{C} \approx 63.2°\text{C}$$

(d) The hot water is a finite heat source. Raising the cold water to 100°C would require extracting all the thermal energy from the hot water, leaving it at 0°C. This violates the Second Law of Thermodynamics, as heat cannot spontaneously flow from a colder body to a hotter one to complete the process.

Question 03

Ice-olation

Suppose that Finn is stranded in the middle of a frictionless ice lake, on a cart which is at rest. Bolted to the cart is a snow machine and a cylindrical tank full of snow. Finn's snow machine is connected to the tank of snow, and begins to expel snow continuously at speed u. We call Finn, the cart, and the bolting mechanisms the system. Assume that the snow machine can eventually empty the tank. We neglect air resistance.

(a) Let the total initial mass of the system be M_i. The snow has mass density ρ. Let the diameter and height of the tank be D and L, respectively. Determine the combined mass of the system once the tank is emptied. Express in terms of M_i, D, ρ, and L, and any physical constants.

(b) The snow machine is initially oriented so that the snow expels parallel to the ground. Determine the final speed of the cart, v_f, when the tank is empty. You may use the following formula, commonly known as the rocket equation, which assumes no external forces:

$$\Delta v = v_e \ln \frac{m_0}{m_f}$$

Δv is the change in speed, v_e is the exhaust speed, m_0 is the initial mass of the system, and m_f is the final mass of the system. Express your answer in terms of u, ρ, D, M_i and L.

(c) Hypothetically, if the snow machine now expels snow at twice the rate of mass per second and with half of the amount of initial snow in the tank, starting from rest, will Finn go faster, slower, or the same speed as your answer in part (b) when the tank is empty?

(d) Finn reaches walkable land, however he has to wait for the cart to come to a stop before exiting. If the coefficient of friction between the cart and the walkable land is μ, how long does it take for the cart to come to a stop? Express your answer in terms of u, μ, M_i, D, L, and any physical constants.

Solution

(a) The snow mass in the cylindrical tank is given by

$$m_s = \rho V = \rho \left(\pi \frac{D^2}{4} \right) L = \rho \frac{\pi D^2 L}{4}$$

The total initial mass is M_i. Once the tank is empty, the remaining mass is

$$\boxed{M_f = M_i - m_s = M_i - \rho \frac{\pi D^2 L}{4}}$$

(b) Using the Tsiolkovsky rocket equation,

$$\Delta v = u \ln \frac{m_0}{m_f}$$

with $m_0 = M_i$ and $m_f = M_f$, we find the final velocity of the cart when the tank is empty:

$$v_f = u \ln \left(\frac{M_i}{M_f} \right)$$

Substituting the expression from part (a),

$$\boxed{v_f = u \ln \left(\frac{M_i}{M_i - \rho \frac{\pi D^2 L}{4}} \right)}$$

(c) In the original case:

$$M_i = M_f + m_s, \quad v_f = u \ln \left(1 + \frac{m_s}{M_f} \right)$$

In the modified case, the tank has half as much snow ($m_s' = \frac{1}{2} m_s$):

$$v_f' = u \ln \left(1 + \frac{m_s'}{M_f} \right) = u \ln \left(1 + \frac{m_s}{2M_f} \right)$$

Since $\ln(1+x)$ increases with x, and $\frac{m_s}{2M_f} < \frac{m_s}{M_f}$, we have $v_f' < v_f$. Therefore, Finn will be slower in this case. Note that expelling the mass quicker does not increase final speed unless the speed of the snow expelled is increased as well. So only the total expelled mass (mass ratio) matters.

$$\boxed{\text{Finn will go slower.}}$$

(d) Once Finn reaches land, friction slows the cart to rest. The frictional force is $F = \mu M_f g$, giving a constant deceleration

$$a = -\mu g$$

Using kinematic equation

$$v = v_i + at,$$

, the time for the cart to stop is

$$t_{\text{stop}} = \frac{v_f}{\mu g}$$

Substituting $v_f = u \ln \left(\frac{M_i}{M_f} \right)$,

$$\boxed{t_{\text{stop}} = \frac{u}{\mu g} \ln \left(\frac{M_i}{M_i - \rho \frac{\pi D^2 L}{4}} \right)}$$

If instead an answer is given in terms of v_f,

$$\boxed{t_{\text{stop}} = \frac{v_f}{\mu g}}$$

Answer

(a) $M_f = M_i - \rho\frac{\pi D^2 L}{4}$

(b) $v_f = u \ln\left(\frac{M_i}{M_i - \rho\frac{\pi D^2 L}{4}}\right)$

(c) Slower; the final speed depends on exhaust speed and mass ratio, not the rate at which the snow is expelled.

(d) $t_{\text{stop}} = \frac{u}{\mu g} \ln\left(\frac{M_i}{M_i - \rho\frac{\pi D^2 L}{4}}\right)$

Question 04

Escape Room: Cosmic Edition

In 1783, the English philosopher and clergyman John Michell wondered if there could be a star massive enough such that no light could escape it, rendering it invisible. Today, we know with certainty that these dark stars, now called black holes, exist, and that they are created when stars collapse under their own gravity at the end of their life cycle.

(a) One example of a star that has a likely future as a black hole is Betelgeuse. Consider Betelgeuse to be spherical with mass M and radius r. For an object of mass m launched from the surface, what is the escape velocity v_e?

+2 correct

(b) The Schwarzschild radius, r_s, is the maximum radius a body of mass M can have before light itself cannot escape, $v_e = c$. Suppose that, after a supernova, Betelgeuse leaves behind a remnant with a mass of $15M_\odot$, or 15 times the mass of the Sun. By what factor must the radius of this remnant be reduced relative to the current radius of Betelgeuse, $r_B = 6 \times 10^8$ km, for it to become a black hole? *Use* $M_\odot = 2 \times 10^{30}$ *kg*

.

(c) Michell realized that black holes could be detected if they were part of a binary system. Imagine a system comprising a black hole with mass M_d and a visible star with mass m_v, where $M_d \gg m_v$ such that the orbit is approximately centered around the black hole. Derive an expression for M_d in terms of the orbital radius r, period T, and the gravitational constant G.

(d) As a photon climbs out of the gravitational well of the star, it loses energy. Explain why an observer at a great distance would see the light shift toward the red end as the star's radius r approaches r_s.

Solution

(a) Using conservation of energy for an escaping mass:

$$U + K = 0$$

Solving for v_e:

$$\frac{1}{2}mv_e^2 = \frac{GMm}{r} \implies v_e^2 = \frac{2GM}{r}$$

$$\boxed{v_e = \sqrt{\frac{2GM}{r}}}$$

Rubric:

2/2 correct

(b) Setting $v_e = c$ to find the Schwarzschild radius r_s:

$$c = \sqrt{\frac{2GM}{r_s}} \implies r_s = \frac{2GM}{c^2}$$

Given $M = 15M_\odot = 15(2 \times 10^{30}) = 3 \times 10^{31}$ kg:

$$r_s = \frac{2(6.67 \times 10^{-11})(3 \times 10^{31})}{(3 \times 10^8)^2} = \frac{4.002 \times 10^{21}}{9 \times 10^{16}} \approx 4.44 \times 10^4 \text{ m} = 44.4 \text{ km}$$

The compression factor is the ratio of the original radius r_B to r_s:

$$\text{Factor} = \frac{6 \times 10^{11} \text{ m}}{4.44 \times 10^4 \text{ m}} \approx 1.35 \times 10^7$$

$$\boxed{\text{Compression Factor} \approx 1.35 \times 10^7}$$

Rubric:

2/2 correct

1/2 finds correct value for r_s

(c) In a stable circular orbit where $M_d \gg m_v$,

$$F_g = F_c \implies \frac{GM_d m_v}{r^2} = \frac{m_v v^2}{r}$$

Substituting the orbital velocity $v = \frac{2\pi r}{T}$:

$$\frac{GM_d}{r^2} = \frac{1}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 r}{T^2}$$

Solving for M_d:

$$\boxed{M_d = \frac{4\pi^2 r^3}{GT^2}}$$

Rubric:

3/3 correct answer

(d) As a photon climbs out of a deep gravitational potential well, it must "work" against the potential field. Since the speed of light c is constant, the total energy of the photon ($E = hf$) must decrease. Because energy is directly proportional to frequency, the frequency f drops. This corresponds to an increase in wavelength ($\lambda = c/f$), shifting the light toward the red end of the spectrum. As $r \to r_s$, the energy required to escape approaches the total energy of the photon, causing the observed frequency to approach zero (an infinite redshift).

Answer

(a) $v_e = \sqrt{\frac{2GM}{r}}$

(b) $r_s \approx 44.4$ km; Compression factor $\approx 1.35 \times 10^7$

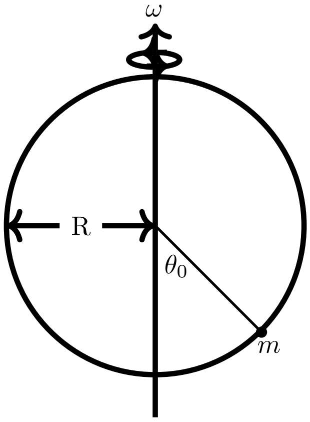
(c) $M_d = \frac{4\pi^2 r^3}{GT^2}$

(d) Energy is lost climbing the potential, so frequency decreases meaning redshift occurs.

Question 05

Bead it!

A bead of mass m is attached to a ring of radius R and mass M. The ring is rotating around a rotation axis along its diameter with a constant angular velocity ω. The ring and rotation axis are in the same plane. Assume that the bead is acted upon by gravity mg, and that there is no friction between the bead and the ring. Let the angle $\theta = 0$ on the ring be defined as the bottom-most point.



(a) If the bead starts off very close to the bottom of the ring, and $\omega \geq \sqrt{g/R}$, what is the stable equilibrium angle θ_0?

(b) In the rotating reference frame of the bead, there exists an additional "force" felt by the bead. Write down this effective potential energy in terms of ω, g, R as a function of θ.

(c) The bead will tend to oscillate around the point you found in part (a). What is the frequency of this small oscillation? On small timescales, the effect of friction is negligible.

Solution

(a)

By balancing the forces in the tangential direction:

$$\begin{aligned} mR\ddot{\theta} &= -mg\sin\theta + m\omega^2(R\sin\theta)\cos\theta \\ \tan\theta_0 &= \frac{mR\sin\theta_0\omega^2}{mg} \\ \frac{1}{\cos\theta_0} &= \frac{R\omega^2}{g} \\ \theta_0 &= \boxed{\arccos\frac{g}{R\omega^2}} \end{aligned}$$

(b)

$$\begin{aligned} U_{\text{eff}}(\theta) &= \text{Gravitational Potential Energy} - \text{Centrifugal Potential Energy} \\ &= \boxed{mgR(1-\cos\theta) - \frac{1}{2}m\omega^2R^2\sin^2\theta} \end{aligned}$$

(c) Using the effective potential and the angle of the bead from part (b), we can show the approximately quadratic energy valley in either $\theta_0 = \pm\arccos\frac{g}{\omega^2 R}$. Let $\theta = \theta_0 + \delta$ with $|\delta| \ll 1$. The small-oscillation frequency satisfies

$$\Omega^2 = \frac{1}{mR^2}U''_{\text{eff}}(\theta_0)$$

With $U_{\text{eff}}(\theta) = mgR(1-\cos\theta) - \frac{1}{2}m\omega^2R^2\sin^2\theta$,

$$U'_{\text{eff}}(\theta) = mR\sin\theta\,(g - \omega^2R\cos\theta), \qquad U''_{\text{eff}}(\theta) = mR\big[g\cos\theta - \omega^2R\cos(2\theta)\big]$$

At equilibrium $\cos\theta_0 = \frac{g}{\omega^2 R}$ (for $\omega^2 \geq g/R$). Substituting in this expression gives:

$$U''_{\text{eff}}(\theta_0) = m\left(\omega^2 R^2 - \frac{g^2}{\omega^2}\right)$$

Therefore,

$$\Omega^2 = \frac{U''_{\text{eff}}(\theta_0)}{mR^2} = \omega^2 - \left(\frac{g}{\omega R}\right)^2$$

$$\boxed{\Omega = \sqrt{\omega^2 - \left(\frac{g}{\omega R}\right)^2}}$$

Note: if $\omega^2 < g/R$, then there is no non-zero equilibrium angle, and $\theta_0 = 0$ remains the stable equilibrium!

Answer

(a)

$$\theta_0 = \arccos\frac{g}{R\omega^2}$$

(b)

$$U_{\text{eff}}(\theta) = mgR(1-\cos\theta) - \frac{1}{2}m\omega^2 R^2 \sin^2\theta$$

(c)

$$\Omega = \sqrt{\omega^2 - \left(\frac{g}{\omega R}\right)^2}$$

Question 06

Resistance is Futile

A highly elastic, perfectly circular loop of conductive wire is placed in a uniform magnetic field B that points perpendicular to the plane of the loop. The wire has a constant total volume V and is made of a material with uniform resistivity ρ. A system stretches the loop outward, increasing its radius at a constant rate v; the loop radius as a function of time is $r(t) = r_0 + vt$.

(a) As the wire stretches, it gets thinner and longer, but its total volume V remains constant. Find the electrical resistance of the loop $R(t)$ as a function of its radius $r(t)$. Make the assumption that the radius is much smaller than the circumference of the ring.

(b) At a high level, Faraday's Law of Induction states that when the total magnetic flux through a loop (magnetic field perpendicular to the loop · area of the loop) changes, a current with magnitude proportional to the flux is generated in the loop. Lenz's Law states that the direction of the magnetic field generated by this current has the effect of opposing the change of the magnetic flux. Conceptually, why must this be the case?

(c) In this problem, the increasing area of the wire loop leads to a change in magnetic flux through it, thereby inducing a current. After performing the full Faraday's Law calculation, one can find that the effective voltage driving the induced current is $\mathcal{E} = 2\pi Bvr(t)$. Calculate the magnitude of the induced current $I(t)$ in the loop. Does the current increase, decrease, or stay the same as the loop expands over time?

(d) Calculate the magnitude of the total magnetic force acting on the wire loop. Does this force assist the system in stretching the wire, or oppose it?

Solution

(a) The resistance of a wire is given by $R = \rho l / A_{\text{cross}}$. The length of the wire is the circumference of the loop, $l = 2\pi r(t)$. Because the volume V is constant, the cross-sectional area is

$$A_{cross} = \frac{V}{l} = \frac{V}{2\pi r(t)}$$

Plugging these in the resistance formula:

$$R(t) = \rho \frac{2\pi r(t)}{\frac{V}{2\pi r(t)}} = \frac{4\pi^2 \rho r(t)^2}{V}$$

3pt for the correct answer

(b) If the induced current generated a magnetic flux in the same direction as the external change, or simply allowed this change to continue without opposing it, the energy of the magnetic field in this direction would increase indefinitely or arbitrarily vanish–which is unphysical. More specifically, if the area of flux was changing unchecked, the magnetic field would occupy arbitrarily more or less space, eventually generating infinite magnetic energy (in the case of increasing area) or allowing the initial magnetic energy to disappear without transferring anywhere (in the case of decreasing area). If the magnetic field strength increased or decreased without bound, there would similarly be infinitely increasing or suddenly disappearing magnetic energy density. The opposing flux generated by the induced current serves to conserve energy by keeping the magnetic flux constant as it tries to change, either by opposing changes in the loop size or magnetic field magnitude (related to part d).

This opposition is required by conservation of energy: otherwise the induced current would reinforce the changing flux and extract unlimited energy from the changing magnetic field without an external energy source.

(c) By Ohm's Law, the induced current is $I(t) = \frac{\mathcal{E}(t)}{R(t)}$

$$I(t) = \frac{2\pi B v r(t)}{\frac{4\pi^2 \rho r(t)^2}{V}} = \frac{BvV}{2\pi \rho r(t)}$$

Because $r(t)$ is in the denominator and the radius is expanding over time, the current **decreases**

(d) The magnetic force on a segment of current-carrying wire is $dF = I dl B$. Since the field is perpendicular to the loop, the total force integrated around the circumference $l = 2\pi r(t)$ is:

$$F_{mag} = I(t) \cdot (2\pi r(t)) \cdot B$$

Plugging in $I(t)$ from part (c):

$$F_{mag} = \left(\frac{BvV}{2\pi \rho r(t)} \right) (2\pi r(t)) B = \frac{B^2 v V}{\rho}$$

By Lenz's Law, the induced current creates a magnetic field that attempts to keep the original flux constant. Since the flux is increasing (loop is expanding), the induced force will try to shrink the loop. Therefore, it **opposes** the mechanical stretching system.

Answer

(a)

$$R(t) = \frac{4\pi^2 \rho r(t)^2}{V}$$

(b) Flux generated in the same direction would lead to infinite energy generation

(c)

$$I(t) = \frac{BvV}{2\pi \rho r(t)}$$

The current decreases as the loop expands.
(d)

$$F_{mag} = \frac{B^2 v V}{\rho}$$

The force opposes the mechanical stretch.

Question 07

Neutri-no Way!

The CUORE experiment is attempting to detect Neutrinoless Double Beta Decay ($0\nu\beta\beta$) using an array of nearly 1000 TeO_2 crystals. Each crystal is cooled to about 10 mK, and attempts to detect a tiny temperature rise caused by energy deposits released during the decay. Neutrinoless Double Beta Decay, if observed, would show that neutrinos are their own antiparticles. Cool!

(a) Nuclear decays often release energies on the scale of 1 MeV. The conversion is $1\text{eV} = 1.602 \times 10^{-19}\text{J}$. Classically, to the nearest order of 10, to what speed would a proton be accelerated to from rest, if it had all the energy from one of these decays? A proton has mass around 1 amu, where $1\text{amu} = 931.5\text{MeV}/c^2$

(b) When a neutral Sodium-22 isotope undergoes typical beta-plus decay, the end products are a neutral Neon-22 isotope, the remaining electron from the Sodium-22, a positron (which has the same mass as an electron), and a neutrino (which is nearly massless). If the electron mass is $5.48 \cdot 10^{-4}$ amu, the mass of neutral Sodium-22 is 21.994 amu, and the mass of neutral Neon-22 is 21.991 amu, how much energy is released by this process?

(c) Each TeO_2 crystal in the CUORE detector acts as a bolometer, also known as a heat detector. When an energy of $E = 2.528$ MeV is deposited, the expected amount of energy released for ^{130}Te double-beta decay, the temperature of the crystal rises by $\Delta T = E/C$, where C is the heat capacity. Assuming that $C = 2 \cdot 10^{-9}$ J/K, calculate the resulting temperature rise.

(d) Explain briefly why operating at millikelvin temperatures makes this measurement feasible.

(e) In theories of $0\nu\beta\beta$, the inverse half-life is related to the effective Majorana neutrino mass $m_{\beta\beta}$ by

$$\frac{1}{T_{1/2}^{0\nu}} = G^{0\nu}|M^{0\nu}|^2 \frac{m_{\beta\beta}^2}{m_e^2}$$

CUORE reports $T_{1/2}^{0\nu} > 3 \cdot 10^{25}$ yr, while another experiment achieves $T_{1/2}^{0\nu} > 3 \cdot 10^{26}$ yr. By what factor does the stronger result improve the upper bound on $m_{\beta\beta}$?

(f) Identify 2 distinct physical mechanisms that could mimic a signal event, and describe one realistic method that CUORE uses (or could use!) to suppress each.

Solution

(a) Using $K = \frac{1}{2}mv^2$ and $m_pc^2 \approx 931.5$ MeV,

$$\frac{v}{c} = \sqrt{\frac{2K}{m_pc^2}} = \sqrt{\frac{2(1\text{ MeV})}{931.5\text{ MeV}}} \approx 4.6 \times 10^{-2}$$

Thus

$$v \approx (4.6 \times 10^{-2})(3.0 \times 10^8) \approx 1.4 \times 10^7\ \text{m/s}$$

To the nearest order of ten:

$$\boxed{10^7\ \text{m/s}}$$

(b) For beta-plus decay using neutral atomic masses,

$$Q = (M_{\text{Na}} - M_{\text{Ne}} - 2m_e)\,c^2$$

The factor of $2m_e$ appears because the neutral daughter atom has one fewer electron and the decay also creates a positron. Therefore

$$\begin{aligned} Q &= \big(21.994 - 21.991 - 2(5.48 \times 10^{-4})\big)\,(931.5\text{ MeV}) \\ &= (0.001904)(931.5\text{ MeV}) \\ &\approx 1.77\text{ MeV} \end{aligned}$$

Hence

$$\boxed{1.77\text{ MeV}}$$

(c) Convert the deposited energy:

$$E = 2.528 \text{ MeV} \cdot 1.602 \times 10^{-13} \text{ J/MeV} = 4.05 \times 10^{-13} \text{ J}$$

Then

$$\Delta T = \frac{E}{C} = \frac{4.05 \times 10^{-13}}{2.0 \times 10^{-9}} \approx 2.0 \times 10^{-4} \text{ K} = 0.20 \text{ mK}$$

Thus

$$\boxed{2.0 \times 10^{-4} \text{ K}}$$

(d) At millikelvin temperatures, the crystals have extremely small heat capacity and low thermal noise. A MeV-scale energy deposit can therefore produce a measurable temperature pulse instead of being washed out by ordinary thermal fluctuations.

(e) From

$$\frac{1}{T_{1/2}^{0\nu}} \propto m_{\beta\beta}^2,$$

the upper bound scales as

$$m_{\beta\beta} \propto \frac{1}{\sqrt{T_{1/2}^{0\nu}}}$$

Improving the half-life bound by a factor of 10 improves the mass bound by

$$\sqrt{10} \approx 3.2$$

The stronger experiment lowers the allowed upper bound on $m_{\beta\beta}$ by a factor of

$$\boxed{3.2}$$

(f) Two realistic backgrounds are:

- Ambient radioactivity from alpha, beta, or gamma decays in nearby materials. This can be suppressed with radiopure materials, shielding, and event-energy cuts around the signal region.
- Cosmic-ray muons or secondary particles. These can be suppressed by operating underground and using veto or coincidence cuts.

Additional suppression can come from rejecting events that trigger multiple crystals, since a true localized decay should mainly heat one crystal.

Answer

(a) 10^7 m/s

(b) 1.77 MeV

(c) 2.0×10^{-4} K, or 0.20 mK

(d) Low heat capacity and low thermal noise make tiny temperature pulses measurable.

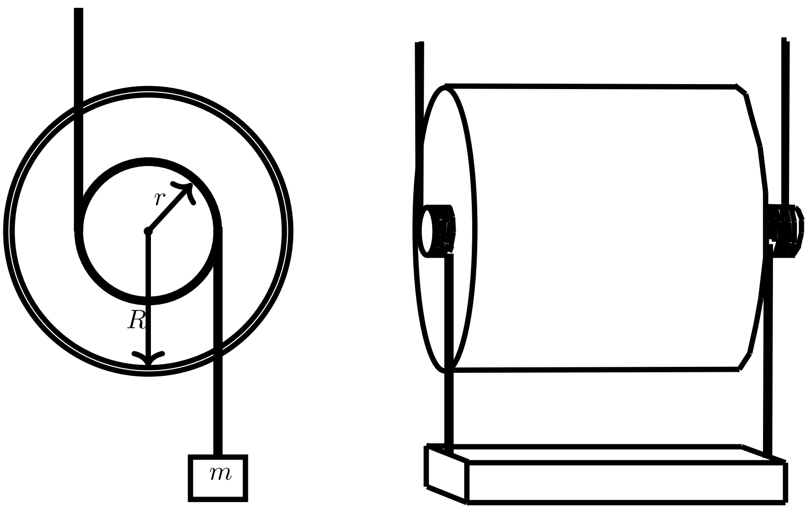
(e) $\sqrt{10} \approx 3.2$

(f) Examples: radioactivity suppressed by shielding/radiopurity, and cosmic rays suppressed by underground operation or vetoes.

Question 08

That's How We Roll

A rigid body of mass m with radius $R = 3r$ is connected to two smaller cylinders of radius r extending out of each end. It is supported by two ideal ropes wrapped around the smaller cylinders, as shown in the diagram. The total moment of inertia about its axis of symmetry is well approximated by $I = mR^2$. A block of the same mass m is supported by two other ropes wrapped around the small cylindrical extensions. The ropes do not slip.



(a) Draw extended free body diagrams for the cylindrical object and the block.

(b) Derive an expression for the magnitude of the angular acceleration α of the cylindrical object as a function of the linear acceleration of the block a_{block} and r.

(c) What is the magnitude of the angular acceleration of the cylinder? Express your answer in terms of r, and use $g \approx 10\,\text{m/s}^2$.

(d) What is the tension in each of the lower ropes that are supporting the block? Express your answer in terms of m and g.

(e) Now consider the same setup, but with the rigid rotating object replaced by one with the same small cylindrical extensions of radius r, but a smaller radius for the central cylinder $R < 3r$, so the moment of inertia is now $mR^2 < 9mr^2$. What condition must R satisfy for the block to accelerate downward with magnitude less than g?

Solution

(a) **Cylindrical object:** A downward force mg from the center of mass. An upward force T_{up} (total tension of the two top ropes) acting at the outer edge of the small cylinder (radius r). A downward force T_{down} (total tension of the two bottom ropes) acting at the opposite edge of the small cylinder (radius r).
Block: A downward force mg from the center of mass. An upward force T_{down} acting on it.

(b) Let the downward acceleration of the cylinder be a_{cyl}. Because it unwinds from the top ropes, $a_{\text{cyl}} = \alpha r$. The block falls due to both the cylinder moving downward AND the rope unwinding from the cylinder. The amount of rope unwound is also accelerating at αr. Therefore, the total acceleration of the block is:

$$a_{\text{block}} = a_{\text{cyl}} + \alpha r = \alpha r + \alpha r = 2\alpha r$$

Solving for α, we get:

$$\alpha = \frac{a_{\text{block}}}{2r}$$

(c) Set up Newton's Second Law equations. For the cylinder (translation):

$$mg + T_{\text{down}} - T_{\text{up}} = ma_{\text{cyl}} = mr\alpha \tag{1}$$

For the cylinder (rotation, noting $I = m(3r)^2 = 9mr^2$ and both tensions cause torque in the same direction):

$$(T_{\text{up}} + T_{\text{down}})r = I\alpha = 9mr^2\alpha \implies T_{\text{up}} + T_{\text{down}} = 9mr\alpha \tag{2}$$

For the block (translation):

$$mg - T_{\text{down}} = ma_{\text{block}} = m(2r\alpha) \implies T_{\text{down}} = m(g - 2r\alpha) \tag{3}$$

Substitute (3) into (2) to find T_{up}:

$$T_{\text{up}} = 9mr\alpha - m(g - 2r\alpha) = 11mr\alpha - mg$$

Substitute T_{up} and T_{down} into (1):

$$mg + (mg - 2mr\alpha) - (11mr\alpha - mg) = mr\alpha 3mg - 13mr\alpha \qquad = mr\alpha \implies 14mr\alpha = 3mg \implies$$

$$\alpha = \boxed{\frac{3g}{14r}}$$

(d) Using equation (3) from the previous part, the total downward tension is:

$$T_{\text{down}} = m\left(g - 2r\left(\frac{3g}{14r}\right)\right) = m\left(g - \frac{3g}{7}\right) = \frac{4mg}{7}$$

Because there are two ropes supporting the block, the tension in *each* rope is half of the total:

$$T_{\text{rope}} = \frac{T_{\text{down}}}{2} = \frac{2mg}{7}$$

(e) We leave $I = mR^2$. The torque equation becomes $T_{\text{up}} + T_{\text{down}} = m\frac{R^2}{r}\alpha$. Adding the translation equations for the cylinder and the block yields $2mg - T_{\text{up}} = 3mr\alpha$, so $T_{\text{up}} = 2mg - 3mr\alpha$. Substituting T_{up} and T_{down} into the torque equation:

$$(2mg - 3mr\alpha) + (mg - 2mr\alpha) = m\frac{R^2}{r}\alpha$$

$$3g - 5r\alpha = \frac{R^2}{r}\alpha \implies \alpha = \frac{3gr}{5r^2 + R^2}$$

The block's acceleration is $a_{\text{block}} = 2r\alpha = \frac{6gr^2}{5r^2+R^2}$. For $a_{\text{block}} < g$:

$$\frac{6gr^2}{5r^2 + R^2} < g \implies 6r^2 < 5r^2 + R^2 \implies r^2 < R^2$$

Since radius must be positive, the condition is $R > r$.

Answer

(a) Free body diagrams (Draw this out)

(b) $\alpha = \frac{a_{\text{block}}}{2r}$

(c) $\alpha = \frac{3g}{14r} \approx \frac{15}{7r}$

(d) $T_{\text{rope}} = \frac{2gm}{7}$

(e) The condition on R for this to occur is $R > r$.

Question 09

POV: You're a Planet Getting Cooked by a Main Sequence Star

A blackbody is an ideal object assumed to absorb and emit radiation perfectly, with no reflection or transmission. A distant star is modeled as a blackbody of radius R_s and temperature T_s, whose total flux I_s and peak wavelength λ_s can be measured from Earth.

A planet of radius R_p orbits the star at radius r. The planet absorbs all incident radiation, has negligible internal heating, and radiates uniformly over its entire surface as a blackbody. Its thermal spectrum peaks at wavelength λ_p. The distance from Earth to the system is d, and you may assume $d \gg r$. You may use the following equations:

$$F = \sigma T^4, \qquad \lambda_{peak} T = b, \qquad L = 4\pi R^2 F, \qquad I = \frac{L}{4\pi d^2}.$$

where F is the flux at the surface of the object, I is the flux at a distance d, L is the luminosity of the object, and σ, b are constants.

(a) Express R_s in terms of I_s, d, λ_s, σ, and b.

(b) Derive an expression for the orbital radius r in terms of I_s, d, λ_p, σ, and b only. Your final answer should not contain R_s, T_s, or λ_s.

(c) Find the ratio of the total flux from the planet to the total flux from the star, both as measured at Earth. Express your answer in terms of R_p and r only.

(d) Considering the ratio found in part (c), briefly explain whether the planet will appear brighter or dimmer if it is closer to the star and why this is consistent with physical intuition.

Solution

(a) From

$$I_s = \frac{L_s}{4\pi d^2} \qquad \text{and} \qquad L_s = 4\pi R_s^2 \sigma T_s^4,$$

we get

$$I_s = \frac{R_s^2 \sigma T_s^4}{d^2}$$

Using Wien's law,

$$T_s = \frac{b}{\lambda_s},$$

so

$$R_s = d\sqrt{\frac{I_s}{\sigma T_s^4}} = d\sqrt{\frac{I_s \lambda_s^4}{\sigma b^4}}$$

Therefore,

$$\boxed{R_s = \frac{d\lambda_s^2}{b^2}\sqrt{\frac{I_s}{\sigma}}}$$

(b) At the planet's orbit, the stellar flux is

$$\frac{L_s}{4\pi r^2}$$

Equating the planet's emitted luminosity to the power it absorbs from the star,

$$\pi R_p^2 \frac{L_s}{4\pi r^2} = 4\pi R_p^2 \sigma T_p^4$$

Canceling R_p^2 and substituting $L_s = 4\pi R_s^2 \sigma T_s^4$ gives

$$T_p = T_s \sqrt{\frac{R_s}{2r}}$$

Using Wien's law for both bodies,

$$\frac{b}{\lambda_p} = \frac{b}{\lambda_s}\sqrt{\frac{R_s}{2r}},$$

so

$$r = \frac{R_s \lambda_p^2}{2\lambda_s^2}$$

Now substitute the result from part (a):

$$r = \frac{1}{2}\left(\frac{d\lambda_s^2}{b^2}\sqrt{\frac{I_s}{\sigma}}\right)\frac{\lambda_p^2}{\lambda_s^2}$$

Hence

$$\boxed{r = \frac{d\lambda_p^2}{2b^2}\sqrt{\frac{I_s}{\sigma}}}$$

(c) Since the planet and star are essentially the same distance from Earth,

$$\frac{I_p}{I_s} = \frac{L_p}{L_s}$$

From the planet's energy balance,

$$L_p = \text{power absorbed from star} = \pi R_p^2 \frac{L_s}{4\pi r^2} = \frac{R_p^2}{4r^2} L_s$$

Therefore,

$$\boxed{\frac{I_p}{I_s} = \frac{R_p^2}{4r^2}}$$

(d) The planet appears brighter when it is closer to the star. Since

$$\frac{I_p}{I_s} \propto \frac{1}{r^2},$$

decreasing r increases the planet-to-star flux ratio. Physically, this is because the planet intercepts more stellar radiation at smaller orbital radius and therefore reradiates more thermal power.

Answer

(a)

$$\boxed{R_s = \frac{d\lambda_s^2}{b^2}\sqrt{\frac{I_s}{\sigma}}}$$

(b)

$$\boxed{r = \frac{d\lambda_p^2}{2b^2}\sqrt{\frac{I_s}{\sigma}}}$$

(c)

$$\boxed{\frac{I_p}{I_s} = \frac{R_p^2}{4r^2}}$$

(d) The planet appears brighter when closer to the star because its absorbed and reradiated power increases as $1/r^2$.